Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021–8928(03)00121-7

J. Appl. Maths Mechs, Vol. 67, No. 5, pp. 737–748, 2003

0021–8928/$—see front matter

INVESTIGATION OF THE SQUEEZING OUT OF THE CURRENT IN SOME FLOWS IN COASTAL WATER-BEARING LAYERS†

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(*Received* 30 *January* 2001)

Mathematical models of certain flows of fresh ground waters, in a semi-infinite pressurized water-bearing layer, to a salt water sea (basin, reservoir, pot hole, etc.), above the surface of which there is a layer of fresh water, are considered within the framework of the two-dimensional theory of steady seepage. To investigate them, mixed boundary-value problems in the theory of analytic functions are formulated and solved using Polubarinova–Kochina's method. On the basis of these models, algorithms are developed for calculating the squeezing out (that is, the process of the forcing out of the seeping fresh waters by the heavier salt waters, leading to deformation of the interface of the liquids) in cases when the ground water flows enter the sea from the side and from below. A detailed analysis of the structure and characteristic features of the processes, as well as of the effect of all the physical characteristics of the models on the nature and degree of the squeezing out of the fresh water, is carried out using the exact analytical relations obtained as well as numerical calculations. In the special case when there is no layer of fresh water above the surface of the sea, a comparison of the results of the calculation is given for both inflow schemes, and the nature of the dependences of the degree of squeezing out of the water from the initial position of contact of the liquids is discussed.

Usually, in problems of the seepage of fresh and saline waters in lenses and borders and, also, under hydrotechnical installations [1–7], the motion of fresh water is considered in strata, the lower part of which is occupied by the heavier, static saline water with an unperturbed surface, which is always horizontal. The existence of horizontal, water-permeable segments in the form of the boundaries of channels, reservoirs, water cases, drains, etc. is also characteristic of these problems. A similar situation is also observed in flow problems in coastal pressurized water-tables [8–16] when the ground water flows enter the sea from below and the profile of the sea bottom is also always assumed to be horizontal.

The problem of the inflow of pressurized ground waters from a horizontal reservoir into a salt water basin [17] is an example of a problem which does not fit into the existing classification [5, 6]. In this case, the initial position of contact between the fresh and saline water is assumed to be vertical. Moreover, vertical equipotentials and an interval of leakage are contained in the flow domain which, in the aggregate, is not entirely characteristic of problems in subterranean hydrodynamics. This leads to a state of affairs where corresponding segments of the boundary to not have a common point of intersection in the plane of the velocity hodograph. Methods based on the Christoffel–Schwartz formula are therefore unsuitable for application to such problems. Furthermore, this domain of the velocity hodograph, as the analysis [1–6] of all possible schemes characteristics of the seepage theory shows, is only encountered in one case, that is, in the classical problem of seepage through a rectangular dam [1, 3, 7, 18]. Polubarinova–Kochina developed an extremely general and effective method [1–7] for investigating this problem, which is based on the use of the analytic theory of linear differential equations of the Fuchs class [19]. The solution of the problem is obtained in the form of integrals of elliptic integrals [20, 21], some of which have logarithmic singularities close to each of the singular points, which gives rise to well-known additional difficulties of a computational type. Hence, despite the apparent simplicity of the scheme for the flow in a coastal water-bearing layer with a lateral discharge, the difficulties in solving the problem are completely analogous to those in the problem of a dam.

In the previous investigations [8–14, 17], essentially no analysis of the effect of the parameters on the flow pattern was carried out.

Both cases (of a lateral inflow and an inflow from below) are investigated below using Polubarinova–Kochina's method. The solution of the problem is initially transformed to a form which is convenient for calculations. In the process, the convergence of all the integrals obtained for the geometrical

†*Prikl. Mat. Mekh.* Vol. 67, No. 5, pp. 836–848, 2003.

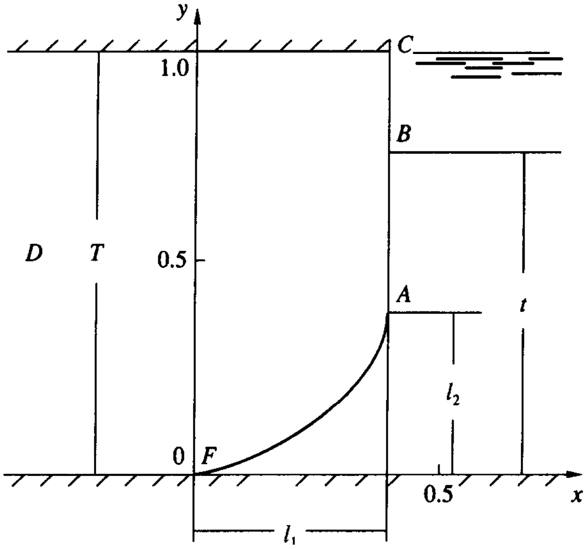

Fig. 1

dimensions and the parameters characterizing the seepage being considered is proved using the arguments put forward by Polubarinova–Kochina [3]. The effect of each physical parameter of the models on the geometrical and seepage characteristics is then analysed using the transformed formulae and numerical calculations, the special features and the degree to which water is squeezed out are studied and, finally, a complete picture of the phenomena is also given. Finally, in the special case of lateral inflow, when there is no layer of fresh water on the sea surface, the results of the calculations are compared for both schemes with the same seepage parameters and the form of the dependences of the degree to which water is squeezed out from the initial position of contact of the liquids is discussed.

1. THE LATERAL INFLOW SCHEME. FORMULATION OF THE PROBLEM

Fresh water with a density ρ_1, moving in a semi-infinite pressurized water table, which is situated on an impermeable bed of rock salt, is squeezed out in the lower part of the layer by heavier stationary salt water of density ρ_2 ($\rho_2 > \rho_1$). At the same time, the initially vertical line of separation between the fresh and salt water in the lower right-hand part of the stratum starts to be deformed, shifting to the left towards the flow, forming a so-called tongue of salt water. Steady motion is possible after a certain, sufficiently long time and, when the brine quietens down, the line of separation turns out to be the streamline for the fresh water [1, 22] and the flow pattern shown in Fig. 1 arises. Under intense exploitation, when the dynamic equilibrium between the fresh and salt waters is disturbed, the threat of penetration of sea water into the water table arises, and the tongue of salt water, on moving in the direction of dry land, can reach the water intake. The determination of the position of the boundary of separation is therefore of great practical interest.

We shall assume that the motion of the ground waters obeys Darcy's law with a known seepage coefficient κ = const. and occurs in homogenous, isotropic earth, which is taken as being incompressible as is the liquid which is seeping through it. The capacity of the water-bearing layer T, the level of the salt water in the sea $t(0 < t \le T)$, the rate of see page Q and the parameter $\rho = \rho_2/\rho_1 - 1$ are assumed to be given. As is usually done in problems of a similar kind [1–14, 17], we will neglect the effect of capillary and diffusion phenomena at the interface of the liquids.

We introduce the complex flow potential $w = \varphi + i\psi$ and the complex coordinate $z = x + iy$, which are dived by κT and T, respectively. It is required to determine the position of the line of separation AF of the domain of seepage z and the pair of harmonic functions φ and ψ, which are conjugate within this domain such that the following boundary conditions are satisfied along the segments of its boundary

$$\begin{aligned} &AB\colon x = l_1, \quad \varphi = \rho(y-t); \quad BC\colon x = l_1, \quad \varphi = 0; \quad CD\colon y = T, \quad \psi = Q \\ &DF\colon y = 0, \quad \psi = 0; \quad AF\colon \psi = 0, \quad \varphi = \rho(y-T) \end{aligned} \tag{1.1}$$

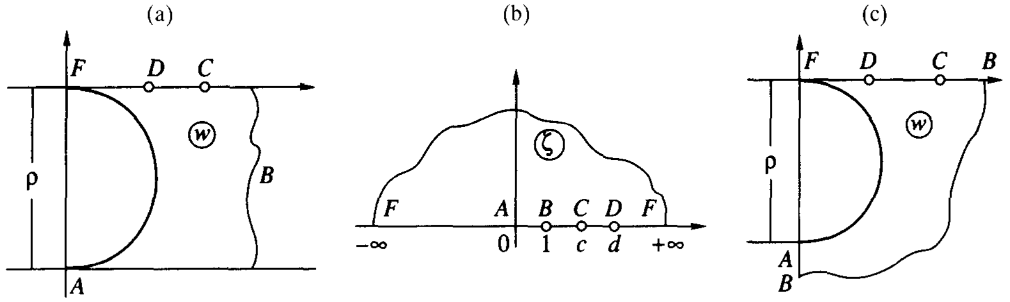

Fig. 2

Here, l_1 and l_2 are the required width and height of the tongue of salt water which has invaded the fresh-water layer.

2. CONSTRUCTION OF THE SOLUTION

We will now consider the domain of the complex velocity w, corresponding to boundary conditions (1.1), which is shown in Fig. 2(a).

This domain, which is a circular triangle, all the angles of which are equal to zero (a modular triangle) is of great significance in the theory of automorphic functions [23, 24]. In seepage theory, a modular triangle is only characteristic in the case of the problem of a rectangular earth dam, the solution of which was obtained for the first time [25, 26] as a solution of a Dirichlet problem and, subsequently, more simply by Polubarinova–Kochina [1, 3, 18]. From the computational point of view, the case considered below is completely analogous to the problem of seepage through an infinitely wide dam [1, p. 78], [3, p. 276]. We emphasize that conformal mapping methods, based on the Christoffel–Schwartz formula, are ineffective when applied to such domains.

To solve the problem, we will use Polubarinova–Kochina's methods, which is based on the use of the analytic theory of linear differential equations. An auxiliary variable ζ is introduced and the functions $z(\zeta)$, which conformally maps the upper half-plane ζ into the domain z (the correspondence of the points is shown in Fig. 2b), the complex velocity $w = d\omega/dz$ and, also

$$Z = dz/d\zeta, \quad F = d\omega/d\zeta \tag{2.1}$$

On determining the characteristics of the functions Z and F near the singular points [6, 8], we find that, in the case in question, they are linear combinations of two branches of the following Riemann function [8, 9]

$$P\begin{bmatrix} 0 & 1 & c & d & \infty & \\ 0 & 0 & -\tfrac{1}{2} & -1 & 2 & \zeta \\ 0 & 0 & -\tfrac{1}{2} & -1 & 2 & \end{bmatrix} = \frac{1}{(d-\zeta)\sqrt{c-\zeta}}P\begin{bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \tfrac{1}{2} & \zeta \\ 0 & 0 & \tfrac{1}{2} & \end{bmatrix} = \frac{Y}{(d-\zeta)\sqrt{c-\zeta}} \tag{2.2}$$

It is clear from relation (2.2) that $\zeta = c$, and $\zeta = d$ are regular points for the function Y, and the linear differential equation of the Fuchs class corresponding to the Riemann symbol (2.2) therefore takes the form

$$\zeta(1-\zeta)Y'' + (1-2\zeta)Y' - \frac{1}{4}Y = 0 \tag{2.3}$$

It is well-known [8, 10] that Eq. (2.3) has two linearly independent integrals

$$Y_1(\zeta) = K(\zeta), \quad Y_2(\zeta) = K'(\zeta) \tag{2.4}$$

which from a fundamental system of solutions in the neighbourhood of the point $\zeta = 0$. Here, $K(\zeta)$ is the complete elliptic integral of the first kind, which is considered as a function of the square of the modulus $k^2 = \zeta, K'(\zeta) = (1-\zeta) = K(k^2), k^2 = 1-\zeta$. Note that $K'(\zeta)$ is a solution containing a logarithmic singularity at the point $\zeta = 0$, close to which the asymptotic representation has the form [27]

$$K'(\zeta) = -\frac{1}{2}\ln\zeta \tag{2.5}$$

The function containing the conformal mapping of the upper half-plane ζ into the domain of the complex velocity w must be expressed in terms of the ratio of linear combinations of the solutions Y_1 and Y_2. If such combinations are constructed and use is made of the correspondence of points A, B and D in the ζ and w plane, we obtain

$$w = \rho(K(\zeta) - iK'(\zeta))/K'(\zeta) \tag{2.6}$$

Taking relation (2.2) and expression (2.6) into account, we find

$$F = iA\rho(K(\zeta) - iK'(\zeta))/\Delta(\zeta), \quad Z = iAK'(\zeta)/(\Delta(\zeta)), \quad \Delta(\zeta) = (d-\zeta)(\sqrt{c-\zeta}) \tag{2.7}$$

where $A > 0$ is an unknown constant. It can be verified that the functions (2.1), which are defined on the basis of relations (2.7), satisfy boundary conditions (1.1), which are formulated in terms of the above-mentioned functions and they are therefore a parametric solution of the initial boundary-value problem.

Writing down representations (2.7) for the different segments of the boundary of the domain ζ, followed by integration along the whole contour of the auxiliary domain, we obtain the closure of the domain of motion z which thereby serves as a check on the calculations. As a result, we obtain the expressions

$$T = \frac{A\pi K'(1/d)}{\sqrt{d(d-c)}}, \quad Q = \frac{A\rho\pi K(1/d)}{\sqrt{d(d-c)}} \tag{2.8}$$

$$T - t = A\int_1^c \frac{K((\zeta-1)/\zeta)}{\sqrt{\zeta}\Delta(\zeta)}d\zeta \tag{2.9}$$

and the coordinates of the points of the line of separation AF

$$x(\zeta) = A\int_{-\infty}^{\zeta} \frac{K(\zeta/(\zeta-1))}{\Delta(\zeta)}d\zeta, \quad y(\zeta) = A\int_{-\infty}^{\zeta} \frac{K(1/(1-\zeta))}{\Delta(\zeta)}d\zeta, \quad -\infty \le \zeta \le 0 \tag{2.10}$$

Putting $\zeta = 0$ in Eqs (2.10) for the coordinates of the points of the line of separation, we find the required dimensions of the tongue

$$l_1 = x(0), \quad l_2 = y(0) \tag{2.11}$$

and, also, the flow rate across the segment BC

$$Q_{BC} = Q - A\rho\int_1^c \frac{K(1/\zeta)}{(d-\zeta)\sqrt{\zeta(c-\zeta)}}d\zeta \tag{2.12}$$

The other expressions for the quantities Q_{BC} and l_2 provide a check on the calculation:

$$Q_{BC} = A\rho\int_0^1 \frac{K(\zeta)}{\Delta(\zeta)}d\zeta, \quad l_2 = t - A\int_0^1 \frac{K'(\zeta)}{\Delta(\zeta)}d\zeta \tag{2.13}$$

3. TRANSFORMATION OF THE FORMULAE TO A FORM CONVENIENT FOR CALCULATIONS

The representations (2.8)–(2.13) contain three unknown constants: A, $c(1 < c < \infty)$ and $d(c < d < \infty)$. The ratio Q/T serves to determine the mapping parameter d, and, from expression (2.8), we obtain

$$K(1/d)/K'(1/d) = Q(\rho T) \tag{3.1}$$

The parameter c is found from Eq. (2.9). Here, the constant A s first eliminated from all of the equations (2.9)–(2.13) by means of the first of relations (2.8) which fixes the capacity T of the water-bearing layer. Relation (3.1) regulates the specification of the physical parameters Q, T and ρ, and, consequently, the domain of applicability of the flow scheme which has been adopted.

As a result of an investigation of system of equations (2.8), (2.9) using the properties of elliptic integrals, it was established that, for a given value of the thickness of the layer t (and, thereby, also of the mapping parameter c) and on fixing two of the three quantities Q, T and ρ, the modulus of the elliptic integrals is uniquely followed from Eq. (3.1). The third physical parameter turns out to be "floating" in this case: the range over which it changes is determined besides, starting from Eq. (3.1) and taking account of the values of Q_*, T_* and ρ_*, which correspond to the cases when $k^2 \approx 0$ and $k^2 \approx 1$. Moreover, for a fixed value of the flow rate Q when the density ρ is reduced and the capacity R is increased by the same factor, the degree of squeezing out increases by the same factor. This behaviour, which is completely natural from the physical point of view, follows directly from relation (3.1) and formulae (2.10): in this case, the right-hand side of Eq. (3.1) and the left-hand sides of Eqs (2.8) are not changed and, consequently, the unknown constant A and d also remain as before. Such an analysis is permissible when one of the three parameters: Q, ρ or T is fixed and the two other parameters are varied in such a way that the ratio $Q/(\rho T)$ does not change. The circumstance considerably extends the range of variation of the input parameters of the model.

The principal computational difficulty with the problem lies in the fact that the integrands in relations (2.9)–(2.13) have, as has already been mentioned, logarithmic singularities in the neighbourhood of the point $\zeta = 0$ and, in addition, they are infinite at the limits of integration. As far as the points $\zeta = c$ and $\zeta = d$ are concerned, it is clear from relations (2.9)–(2.13) that all the integrals are convergent at this point. The finiteness of the quantities l_1 and l_2 and, also, of expression (2.12) and (2.13) for the flow rate then follows from representation (2.5).

For computational convenience, we introduce the notation $\alpha = 1/d$ and $\beta = 1/c$ $(0 < \alpha < \beta \le 1)$ and, following Polubarinova-Kochina [3, p. 278], we replace ζ by corresponding expressions for the different integrals, which make the integrands in expressions (2.9)–(2.13) finite at the integration limits and, in fact, we put

$$\zeta = \sin^2 t \ \text{ when } \ 0<\zeta<1; \quad \zeta = 1-1/\tau, \quad \tau = \sin^2 t \ \text{ when } \ -\infty<\zeta<0$$

$$\zeta = 1/\tau, \quad \tau = \beta+(1-\beta)\sin^2 t \ \text{ when } \ 1<\zeta<c$$

As a result, we arrive at the following computational relations

$$T = \frac{A\pi K'(\alpha)}{2\sqrt{\beta-\alpha}}, \quad Q = \frac{A\rho\pi K(\alpha)}{2\sqrt{\beta-\alpha}}$$

$$T-t = A\sqrt{\beta_1}\int_0^{\pi/2}\frac{K(\beta_1\cos^2 t)\cos t}{\beta-\alpha+\beta_1\sin^2 t}\text{dt}$$

$$x(t) = I_s(t;\alpha_1,\beta_1), \quad y(t) = I_c(t;\alpha_1,\beta_1), \quad 0\le t\le\pi/2$$

$$l_1 = x(\pi/2), \quad l_2 = y(\pi/2)$$

$$Q_{BC} = Q-A\rho\sqrt{\beta_1}\int_0^{\pi/2}\frac{K(\beta+\beta_1\sin^2 t)\cos t}{\beta-\alpha+\beta_1\sin^2 t}\text{dt}$$

$$Q_{BC} = \rho I_s(\pi/2;\alpha,\beta), \quad l_2 = t-I_c(\pi/2;\alpha,\beta)$$

Here,

$$\alpha_1 = 1-\alpha, \quad \beta_1 = 1-\beta$$

$$I_s(t;\alpha,\beta) = A\int_0^t\frac{K(\sin^2 t)\sin t\cos t}{(1-\alpha\sin^2 t)\sqrt{1-\beta\sin^2 t}}\text{dt},$$

$$I_c(t;\alpha,\beta) = A\int_0^1\frac{K(\cos^2 t)\sin t\cos t}{(1-\alpha\sin^2 t)\sqrt{1-\beta\sin^2 t}}\text{dt}$$

Table 1

$\rho \times 10^4$	$l_1 \times 10^4$	$l_2 \times 10^4$	$Q_{BC} \times 10^4$	$T \times 10^3$	$l_1 \times 10^4$	$l_2 \times 10^4$	$Q_{BC} \times 10^4$
65	859	832	71	605	368	367	76
70	1444	1386	61	679	1041	997	60
101	3900	3465	44	997	4106	3615	44
165	8100	5802	39	1662	14352	9982	39
321	17268	7824	38	3268	60405	26012	37

Table 2

$Q \times 10^4$	$l_1 \times 10^4$	$l_2 \times 10^4$	$Q_{BC} \times 10^4$	$t \times 10^3$	$l_1 \times 10^4$	$l_2 \times 10^4$	$Q_{BC} \times 10^4$
28	17974	7905	12	345	2513	2094	75
46	10445	6581	20	550	3671	3122	63
87	4401	3825	41	762	4455	3863	45
130	1666	1593	85	942	4747	4151	19
152	211	206	143	1000	4766	4170	0

When evaluating the integrals, it is possible, by expanding the integrands in power series in the small parameters α and β, to use the well-known formulae [27]

$$\int_0^{\pi/2} K(\sin^2 t)\sin t dt = \frac{\pi}{4}, \quad \int_0^{\pi/2} K(\cos^2 t)\sin t dt = 2G$$

where $G = 0.915966$ is the Catalan constant.

4. CALCULATION OF THE FLOW SCHEME AND ANALYSIS OF THE NUMERICAL RESULTS

The flow pattern, calculated for $T = 1$, $\rho = 0.0105$, $Q = 0.009$ and $t = 0.7623$, is shown in Fig. 1. The results of calculations of the effect of the governing physical parameters ρ, Q, T and t on the magnitudes of l_1, l_2 and Q_{BC} are presented in Tables 1 and 2. In each of the three blocks of the tables (they are separated by vertical lines) one of the above-mentioned parameters is varied (within the permissible range) and the values of the remaining parameters are fixed (the basic version): $T = 1$, $\rho = 0.0105$, $Q = 0.009$ and $t = 0.7623$. The graphs of the quantities l_1 (curve 1) and l_2 (curve 2) as functions of ρ, T, Q and t are shown by the solid lines in Figs 3–6. The dashed lines correspond to the special case when $t = T$ when there is no layer of fresh water over the sea surface.

An analysis of the data in the tables and the graphs leads to the following conclusions.

An increase in the capacity of the water-bearing layer, the thickness of the salt-water layer and the density of the saline waters, and a reduction in the flow rate increase the extent to which the fresh water is squeezed out.

The identical qualitative form of the graphs of l_1 and l_2 as functions of the parameters ρ, T and t is noteworthy: an increase in the density ρ_2 of the salt waters, the capacity of the pressurized layer T and the thickness of the salt-water layer t leads to an increase in the dimensions of the tongue of salt water. For instance, when the parameters ρ is increased by a factor of 4.9, the quantities l_1 and l_2 increase by a factor of 20.1 and 9.4 respectively. On the other hand, an increase in the flow rate Q by a factor of 5.4 leads to a decrease in l_1 and l_2 by a factor of 85.2 and 38.4 respectively.

The capacity of the water-bearing layer is found to have the greatest effect on the extent to which the fresh water is squeezed out. It can be seen that, when T is increased by a factor of 5.4, the quantities l_1 and l_2 increase by a factor of 164 and 73, respectively. At the same time, the height of the tongue can reach 80% of the capacity of the layer.

For all of the blocks of the tables, it is noteworthy that the approximate equality $l_1 \approx l_2$ is satisfied in the case of small values of ρ and T and large values of Q. On the other hand, for large values of ρ and T and small values of Q, we have $l_1 \approx 2.2 l_2$.

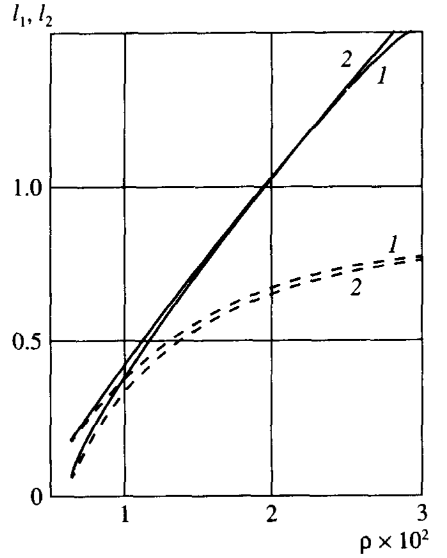

Fig. 3

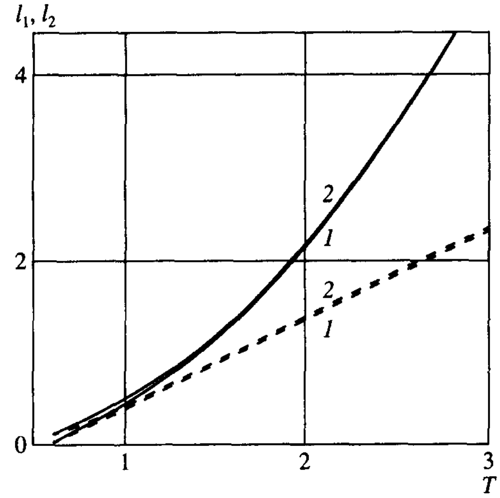

Fig. 4

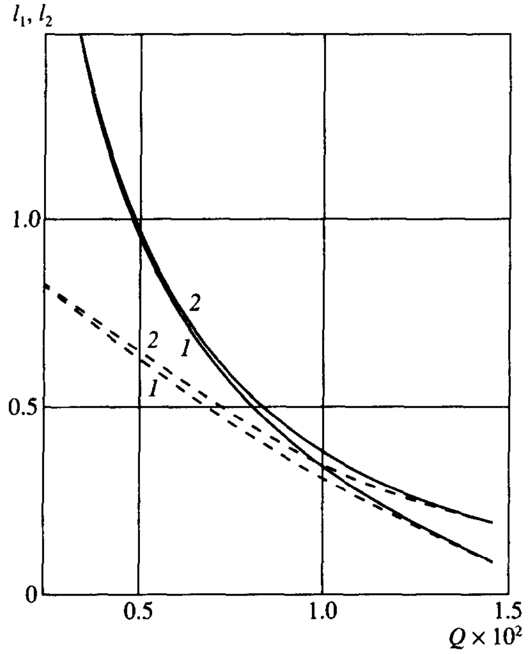

Fig. 5

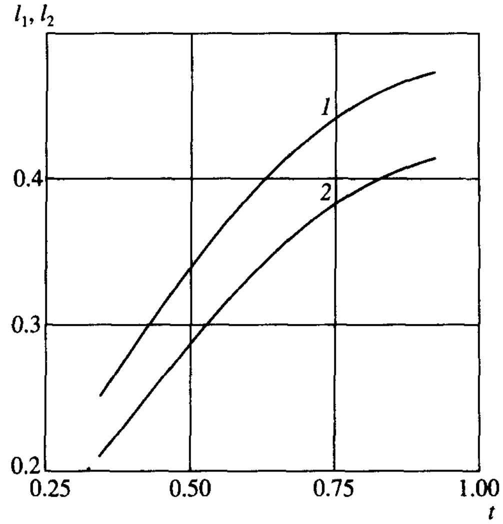

Fig. 6

The behaviour of the dimensions of the tongue as a function of the thickness of the salt-water layer in the sea is of particular interest. It is clear from the graph in Fig. 6 that the dependences of l_1 and l_2 on t are qualitatively similar while, for a fixed value of t, the width of the tongue always exceeds its height by 14–20%.

The flow rate across the segment BC decreases as ρ, T and t are increased. At the same time, while the flow rate Q_{BC} only changes by a factor of two when the parameters ρ and T vary, an increase in the salt-water level leads to a reduction in it by a factor of almost 19. Conversely, increasing the flow rate Q leads to a considerably increase in the flow rate Q_{BC}, which is completely natural from the physical point of view. For instance, when the parameter Q changes from 0.0028 to 0.0152, the flow rate Q_{BC} increases by a factor of almost 12 and, at the same time, the ratio Q_{BC}/Q increases from 0.43 and 0.94.

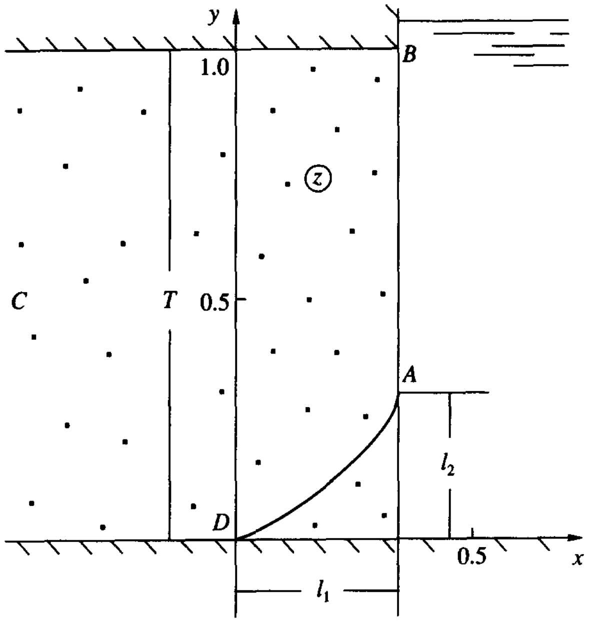

Fig. 7

Table 3

$\rho \times 10^4$	$l_1 \times 10^3$	$l_2 \times 10^3$	$Q \times 10^4$	$l_1 \times 10^3$	$l_2 \times 10^3$	$T \times 10^2$	$l_1 \times 10^3$	$l_2 \times 10^4$
66	180	176	20	1828	802	6	119	1161
	545	−680		2533	100		358	−4486
83	235	227	52	1059	673	83	196	1887
	598	566		1048	260		499	4704
114	454	402	84	489	425	114	516	4566
	711	429		731	412		807	4886
179	885	620	116	257	246	179	1584	11072
	988	279		609	549		1760	4985
417	1621	805	148	166	163	417	7743	33535
	2125	120		550	−668		8847	5000

5. A SPECIAL CASE

In the case when the points B and C merge in the z and ζ planes, which corresponds to parameter values $t = T$ and $c = 1$, we have a flow scheme in which there is no layer of fresh water on the sea surface. The results for this case are obtained from formulae (3.2) when $\beta = 1$, $\beta_1 = 0$. In this case, Eq. (2.9) is transformed into an identity and, from Eqs (2.12) and (2.13), it follows that $Q_{BC} = Q$.

The flow pattern, calculated for $T = 1$, $Q = 0.01$ and $\rho = 0.01$, is shown in Fig. 7. The results of calculations of the effect of the governing physical parameters ρ, Q and T on the dimensions l_1 and l_2 with the basic version $T = 1$, $Q = 0.01$ and $\rho = 0.01$ are shown in Table 3 (the upper rows of value of l_1 and l_2). The graphs of the quantities l_1 and l_2 against ρ, Q and T are shown by the solid lines in Figs 8–10.

It can be seen that the dependence of the height of the tongue l_2 on the capacity of the pressurized layer T turns out to be close to linear. For $0 < T < 5$ and $\rho = Q = 0.01$, we can take $l_2 = 0.95(T - 0.78)$. It is also noticeable that, in the case of the above-mentioned values of T and Q, we have an almost quadratic relation $l_2^2 = 2p\rho$, where $17 \le p \le 20$.

6. THE SCHEME FOR INFLOW FROM BELOW

Formulation of the problem and its solution. The traditional scheme [8–14] for the flow of fresh ground waters into a coastal pressurized water-bearing layer which enters the sea from below is shown in

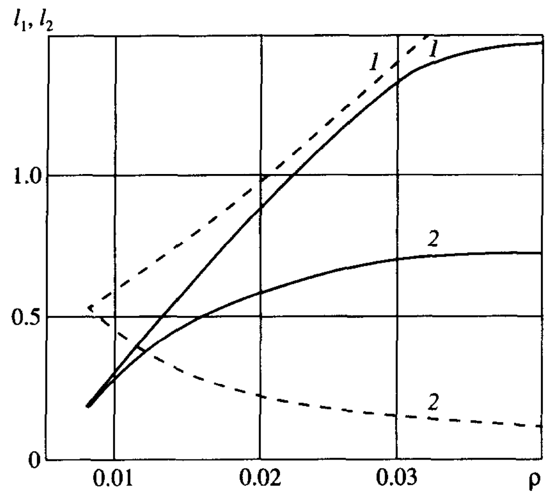

Fig. 8

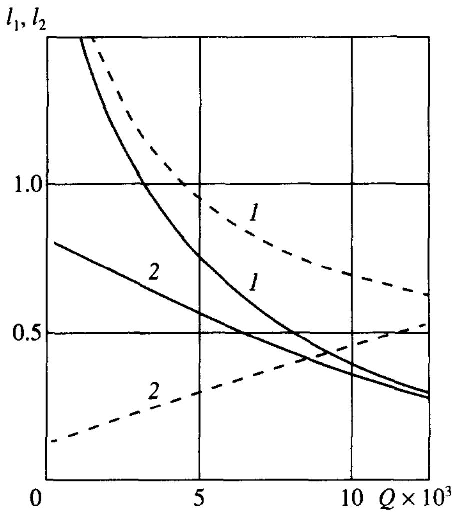

Fig. 9

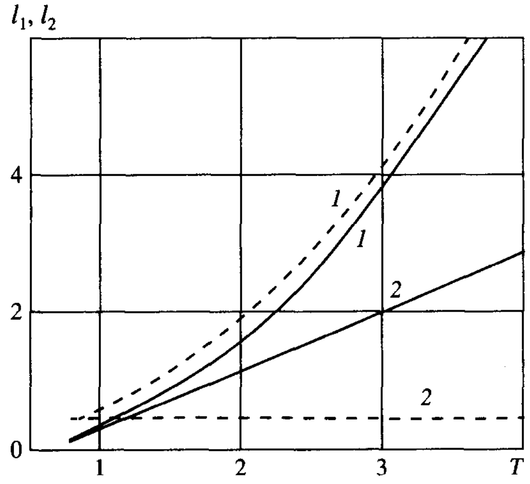

Fig. 10

Fig. 11. The problem reduces to determining the complex potential $\omega(z)$ with boundary conditions (1.1) with the sole difference that the conditions for the segment AB are replaced by the conditions $y = T$, $\varphi = 0$.

The required functions Z and F are determined by the following Riemann symbol

$$P\begin{bmatrix} 0 & 1 & c & \infty & \\ 0 & 0 & -1 & 2 & \zeta \\ -1/2 & -1/2 & -1 & 2 & \end{bmatrix} = \frac{1}{(c-\zeta)\sqrt{\zeta(1-\zeta)}}P\begin{bmatrix} 0 & 1 & \infty & \\ 0 & 0 & 0 & \zeta \\ 1/2 & 1/2 & 0 & \end{bmatrix} = \frac{Y}{(c-\zeta)\sqrt{\zeta(1-\zeta)}} \tag{6.1}$$

The linear differential equation of the Fuchs class corresponding to (6.1)

$$\zeta(1-\zeta)Y'' + \left(\frac{1}{2}-\zeta\right)Y' = 0$$

has two linearly independent integrals

$$Y_1 = \text{const}, \quad Y_2 = \arcsin\sqrt{1-\zeta}$$

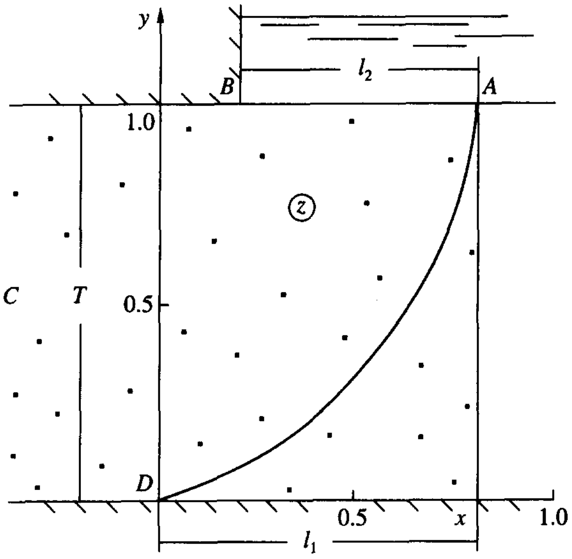

Fig. 11

The conformal mapping of the upper half-plane of the auxiliary parametric variable ζ (Fig. 2b) onto the complex velocity domain (Fig. 2c) has the form

$$w = -\frac{i\pi\rho}{2\arcsin\sqrt{1-\zeta}} \tag{6.2}$$

Taking relation (6.1) and expression (6.2) into account, we find

$$F = \frac{iA}{\Delta(\zeta)}, \quad Z = -\frac{2A}{\pi\rho}\frac{\arcsin\sqrt{1-\zeta}}{\Delta(\zeta)}, \quad \Delta(\zeta) = (c-\zeta)\sqrt{\zeta(1-\zeta)}, \quad A>0 \tag{6.3}$$

The expressions for the governing parameters of the model, corresponding to expressions (2.8) and (2.10), take the form

$$T = \frac{2A\ln(\sqrt{c}+\sqrt{c-1})}{\rho\sqrt{c(c-1)}}, \quad Q = \frac{\pi A}{\sqrt{c(c-1)}} \tag{6.4}$$

and the coordinates of the points of separation AD ($-\infty \le \zeta \le 0$) are

$$x(\zeta) = \frac{2A}{\pi\rho}\int\limits_{-\infty}^{\zeta}\frac{\ln(\sqrt{-\zeta}+\sqrt{1-\zeta})}{\Delta(\zeta)}d\zeta, \quad y(\zeta) = \frac{A}{\rho}\int\limits_{-\infty}^{\zeta}\frac{d\zeta}{\Delta(\zeta)} \tag{6.5}$$

Unlike the preceding scheme and formulae (2.8), in this case formulae (6.4) enable us to express the unknown constants in terms of the governing parameters of the model in the explicit form

$$\alpha = \mathrm{ch}^{-2}\frac{\pi\rho T}{Q}, \quad A = \frac{Q}{2\pi}\mathrm{sh}\frac{\pi\rho T}{Q} \tag{6.6}$$

Transforming formulae (6.5) in the same way as in Section 3, we arrive at the following computational functions, corresponding to formulae (3.2)

$$x(t) = \frac{4A\alpha}{\rho}\int\limits_{t}^{\pi/2}\frac{\cos t}{1-\alpha_1\sin^2 t}\ln\frac{1+\sin t}{\cos t}dt$$

$$y(t) = \frac{A\alpha}{\rho\sqrt{\alpha_1}}\ln\left(\exp\left(\frac{\pi\rho T}{Q}\right)\frac{1-\sqrt{\alpha_1}\sin t}{1+\sqrt{\alpha_1}\sin t}\right) \tag{6.7}$$

$$0 \le t \le \frac{\pi}{2}, \quad \alpha_1 = 1-\alpha = \operatorname{th}^{-2}\frac{\pi\rho T}{2Q}$$

$$l_1 = x(0), \quad l_2 = \frac{4A\alpha}{\pi\rho}\int_0^{\pi/2}\frac{t}{1-\alpha\cos^2 t}dt = \frac{A}{\pi\rho\sqrt{\alpha^{-2}-1}}\left[\frac{\pi^2}{2} - 4\sum_{k=1}^{\infty}\frac{(\alpha^{-1}-\sqrt{\alpha^{-2}-1})^{2k+1}}{(2k+1)^2}\right] \tag{6.8}$$

A well-known result [27, p. 45] has been used here.

Note that, since, in the case being considered, there is a common point of intersection of the boundary segments in the complex velocity domain (Fig. 2c), formulae (6.7) and (6.8) can also be obtained by inversion of the domain w.

7. ANALYSIS OF THE NUMERICAL CALCULATIONS AND COMPARISON OF THE RESULTS FOR THE TWO SCHEMES

The flow pattern, calculated for the same values of the parameters T, Q and ρ as in Fig. 7, is shown in Fig. 11. The results of calculations of the effect of the governing physical characteristics ρ, Q and T on the quantities l_1 and l_2 (negative values of l_2, corresponding to the scheme with inflow from below, indicate that, in the flow plane, point B, that is, the left-hand boundary of the sea bottom, is displaced to the left of the ordinate) are shown in Table 3 (the lower rows). The graphs of the degree of squeezing of l_1 (curves 1) and l_2 (curves 2) against ρ, Q and T are represented by the dashed lines in Figs 8–10.

The analysis of the dependence of the required quantities on the above-mentioned physical parameters reduces to the following.

As in the earlier problem, an increase in the capacity of the layer leads to an enlargement of the tongue. It is noteworthy that the dependence of the width l_1 on T is qualitatively similar here to the case of a lateral inflow (Fig. 10). However, compared with the first scheme, the form of the dependence of the quantity l_2 when the parameters ρ and Q are varied, changes in a radical manner: its growth is now caused by a decrease in the density of the saline waters and an increase in the flow rate (Figs 8 and 9).

As previously, it is the capacity of the layer which is found to have the greatest effect on the width l_1. For instance, when T is increased by a factor of 6.3, the parameter l_1 increases by a factor of 25. The ratios $l_1^{(2)}/l_1^{(1)}$ and $l_2^{(2)}/l_1^{(1)}$, where the superscript indicates a calculation using the first or second scheme, change by factors of 2.5 and 25 respectively for the values of T presented in the table.

It is clear from the graphs presented in Figs 9 and 10 that the dependence of l_2 on Q and T is linear, and one can take $l_2 \approx 0.32(Q - 0.09)$ and $l_2 \approx 0.48$ in the ranges of values of Q and T considered. For small values of ρ and T and large values of Q, the quantities l_1 and l_2 differ by just 2%. While $l_1 < l_2$, that is, the abscissa of point B in the z plane is displaced to the left from the ordinate.

Conversely, in the case of large values of ρ and T and small values of Q, the width l_1 is now greater than the quantity l_2 by a factor of 18 and, when $Q = 0.002$, it is even 25 times greater.

I wish to thank M. G. Khublaryan for critical remarks and his interest.

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Translated by E.L.S.